Table 1 Comparative results

Time interval, days	Measured Q	Initial set of Δn_i	Local minimum Q	Optimal set of Δn_i
2 to 16	9.0209	0.5340	6.4260	0.7402
		1.0486		1.4924
		1.5449		1.8387
		2.0239		2.1739
2 to 32	9.1496	0.5340	8.1946	0.5590
		1.0486		1.1236
		1.5449		1.8949
		2.0239		2.0552
2 to 182	9.8488	0.5340	9.6071	0.6340
		1.0486		1.1486
		1.5449		1.2449
		2.0239		1.6177
2 to 367	9.9967	0.5340	9.8012	0.0840
		1.0486		0.6486
		1.5449		1.1195
		2.0239		1.7239

Since the optimal, but unattainable, Q would be zero (the case of rigid symmetry), a subsequent analysis of the behavior of Q as a function of "launch" parameters and time interval was performed. Using a numerical minimization technique, a local minimum Q was determined by finding the optimal Δn_i for a given $T_2 - T_1$ (see Table 1). It was found that, as the time interval was decreased, the local minimum Q diminished, and a distinct set of optimal Δn_i was generated for each $T_2 - T_1$. When $T_2 - T_1$ was increased to one year, the improvement in Q by minimization was neglible. Hence, any choice of "launch" parameters that insure significant differences in mean angular rate will provide acceptable dispersion over long periods of time.

Rarefied Viscous Flow Near a Sharp Leading Edge

Edgar Bendor*

Republic Aviation Corporation, Farmingdale, N. Y.

Nomenclature

M = freestream Mach number
Re = Reynolds number
s = h_w/H_∞
t = $U_\infty^2/2H_\infty$
β = shock inclination
λ = $\gamma C\mu_w U_\infty/(\gamma + 1)(\gamma - 1)h_w\rho_\infty$
ξ = $(\gamma + 1)Re/\gamma t M^2$
χ = $M^3/(Re)^{1/2}$

In the rarefied flow over a sharp edged plate there arises a so-called "viscous layer" regime, in which the strong pressure interaction theories are no longer applicable, but the degree of rarefaction is not yet such that slip and temperature jump need be taken into account. For a cold wall, the regime in question occurs for values of the viscous interaction parameter χ of the order of the square of the freestream Mach number. Neglecting the free molecule and slip regions in the immediate vicinity of the leading edge, the viscous layer is bounded by a (thin) shock wave, originating at the leading edge, behind which the entire flow field is viscous.

Such a model has been treated by Oguchi,[1] using the boundary layer equations, with the assumption that the curvature of the shock wave is zero at the leading edge. Having shown that shear stress and heat conduction are predominant at the leading edge, he simply equates these terms to zero. The basic (vanishing Reynolds number) solution so obtained is modified later to include terms of the next order, but with the downstream coordinate treated as a parameter rather than as an independent variable.

It is shown here that Oguchi's solution may be regarded as the zeroth term of a series solution of the full boundary layer equations in powers of the variable $\xi = [2(\gamma + 1)H_\infty/\gamma U_\infty^2](Re/M^2)$ of which the first three terms are derived. The shock angle at the leading edge is found to be $\beta = [(\gamma + 1)(\gamma - 1)/12\gamma]^{1/2}$ for a cold wall. The validity of the boundary layer approximation in a region with this apex angle is open to question (e.g., Hayes and Probstein[2]). It is apparent, however, that a solution cannot be obtained easily to more general equations, and the fact that the pressure at the leading edge is found to be in good agreement with experiment provides some justification for the following procedure.

If the boundary layer equations for unit Prandtl number are transformed with

$$\eta = \left(\frac{U_\infty}{2x}\right)^{1/2} \int_0^y \rho dy \qquad (1)$$

$$f = \int_0^\eta \left(\frac{u}{U_\infty}\right) d\eta \qquad \Theta = \frac{H - h_w}{H_\infty - h_w} \qquad (2)$$

then the momentum and energy equations are

$$\frac{\partial}{\partial\eta}\left(\rho\mu \frac{\partial^2 f}{\partial\eta^2}\right) + f\frac{\partial^2 f}{\partial\eta^2} = 2x\left[\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial x\partial\eta}\right) - \left(\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial\eta^2}\right)\right] + \frac{2x}{\rho U_\infty^2}\frac{dp}{dx} \qquad (3)$$

$$\frac{\partial}{\partial\eta}\left(\rho\mu \frac{\partial\Theta}{\partial\eta}\right) + f\frac{\partial\Theta}{\partial\eta} = 2x\left[\left(\frac{\partial f}{\partial\eta}\frac{\partial\Theta}{\partial x}\right) - \left(\frac{\partial f}{\partial x}\frac{\partial\Theta}{\partial\eta}\right)\right] \qquad (4)$$

Neglecting the quantity $(U_\infty - u_s) \sim U_\infty O(\beta^2)$, the boundary conditions are

$$\eta = 0 \qquad f = \partial f/\partial\eta = \Theta = 0$$
$$\eta = \eta_s \qquad \partial f/\partial\eta = \Theta = 1 \qquad (5)\dagger$$

Continuity requires that

$$\rho_\infty U_\infty y_s = \psi_s = (2U_\infty x)^{1/2} f(x,\eta_s) \qquad (6)$$

But from Eq. (1)

$$y_s = \left(\frac{2x}{U_\infty}\right)^{1/2} \int_0^{\eta s} \frac{1}{\rho} d\eta \qquad (7)$$

and for an ideal gas

$$1/\rho \approx [(\gamma - 1)H_\infty/\gamma p][(1 - s)\Theta + s - t(\partial f/\partial\eta)^2] \qquad (8)$$

Substituting Eqs. (7) and (8), the continuity condition (6) becomes

$$\left[(\gamma - 1)\frac{\rho_\infty H_\infty}{\gamma}\right]\int_0^{\eta_s}\left[(1 - s)\Theta + s - t\left(\frac{\partial f}{\partial\eta}\right)^2\right] \times d\eta = p(x)f(x,\eta_s) \qquad (9)$$

The outer boundary conditions, which are satisfied at the shock rather than at infinity, suggest transformation to the variable $\eta/\eta_s(x)$. Further, since the shear stress is known to become predominant for vanishing Reynolds number, one may expect the leading term in an expansion of $u(x,\eta)$ to be

Received by ARS October 29, 1962; revision received February 18, 1963.

* Specialist Engineer, Applied Research and Development Division.

† Subscript s refers to conditions just behind the shock wave.

given by $(\partial/\partial y)(\mu\partial u/\partial y) = 0$ or $u \sim \text{func}(x)\int(1/\mu)dy$. Hence, with

$$\mu = CT\,\mu_w/T_w \tag{10}$$

one obtains simply $u/U_\infty = \eta/\eta_s(x)$. If, therefore, the transformations

$$\eta/\eta_s(x) = Y \qquad f(x,\eta) = \eta_s(x)F(x,Y) \tag{11}$$

are introduced, and Eqs. (8) and (10) substituted, the momentum, energy, and continuity equations become

$$\frac{\gamma}{\gamma-1}C\frac{\mu_w}{h_w}p\frac{1}{\eta_s^2}\frac{\partial^3F}{\partial Y^3} + \frac{1}{\eta_s^2}\frac{d}{dx}(x\eta_s^2)F\frac{\partial^2F}{\partial Y^2} = 2x\left[\left(\frac{\partial F}{\partial Y}\frac{\partial^2F}{\partial x\partial Y}\right) - \left(\frac{\partial F}{\partial x}\frac{\partial^2F}{\partial Y^2}\right)\right] + \frac{\gamma-1}{\gamma}\frac{1}{t}\times \left[(1-s)\Theta + s - t\left(\frac{\partial F}{\partial Y}\right)^2\right]\frac{x}{p}\frac{dp}{dx} \tag{12}$$

$$\frac{\gamma}{\gamma-1}C\frac{\mu_w}{h_w}p\frac{1}{\eta_s^2}\frac{\partial^2\Theta}{\partial Y^2} + \frac{1}{\eta_s^2}\frac{d}{dx}(x\eta_s^2)F\frac{\partial\Theta}{\partial Y} = 2x\left[\left(\frac{\partial F}{\partial Y}\frac{\partial\Theta}{\partial x}\right) - \left(\frac{\partial F}{\partial x}\frac{\partial\Theta}{\partial Y}\right)\right] \tag{13}$$

and

$$\frac{\gamma-1}{\gamma}\rho_\infty H_\infty\int_0^1\left[(1-s)\Theta + s - t\left(\frac{\partial F}{\partial Y}\right)^2\right]\times dY = p(x)\,F(x,1) \tag{14}$$

respectively. The boundary conditions are

$$\begin{array}{ll} Y = 0 & F = \partial F/\partial Y = \Theta = 0 \\ Y = 1 & \partial F/\partial Y = \Theta = 1 \end{array} \tag{15}$$

Finally, in the hypersonic approximation $(M\beta)^2 \gg 1$, the pressure is given by

$$\begin{aligned} p(x) &= [2\gamma/(\gamma+1)]p_\infty M^2[(d/dx)(\psi_s/\rho_\infty U_\infty)]^2 \\ &= [2U_\infty/(\gamma+1)\rho_\infty]\{(d/dx)[(2x)^{1/2}\eta_s(x)\,F(x,1)]\}^2 \end{aligned} \tag{16}$$

The boundary conditions require that the leading terms in the expansions of $F(x,Y)$, $\Theta(x,Y)$ be independent of x. Hence from Eq. (16), $\eta_s(x) \sim (x)^{1/2}$ as $x \to 0$ if the pressure is to be finite at the leading edge. On the other hand, if the pressure gradient is to vanish as $x \to 0$, $\eta_s(x)/(x)^{1/2}$ and $F(x,Y)$ must be expressible as series in powers of x^s with $s > 1$. But such a solution can be shown to be inconsistent with Eqs. (12) and (13). One may conclude, therefore, that the leading edge cannot be an inflexion point of the shock shape, and that there remains a finite pressure gradient at the leading edge.

The appropriate series are

$$\eta_s(x) = ax^{1/2}(1 + b_1x + b_2x^2 + \ldots) \tag{17}$$

$$F(x,Y) = F_0(Y) + F_1(Y)x + F_2(Y)x^2 + \ldots \tag{18}$$

$$\Theta(x,Y) = \Theta_0(Y) + \Theta_1(Y)x + \Theta_2(Y)x^2 + \ldots \tag{19}$$

and the boundary conditions are

$$\begin{array}{ll} F_r(0) = \Theta_r(0) = F_r'(0) = 0 & (r = 0, 1, 2, \ldots) \\ F_0'(1) = \Theta_0(1) = 1 & \\ F_r'(1) = \Theta_r(1) = 0 & (r = 1, 2, 3, \ldots) \end{array} \tag{20}$$

If Eqs. (12) and (13) are multiplied by η_s^2p and η_s^2, respectively, and the forementioned series substituted, it is noted that $F_r(Y)$, $\Theta_r(Y)$ appear as coefficients of x^r only in the shear stress and heat conduction terms. The required functions therefore are derived by integration, rather than as solutions of sets of differential equations, and the boundary conditions are sufficient for their complete determination. The result is

$$F_0 = Y^2/2 \qquad \Theta_0 = Y \tag{21}$$

$$120\lambda F_1 = 5Y^2 - 2Y^5 \qquad 12\lambda\Theta_1 = Y - Y^4 \tag{22}$$

$$7200\lambda^2F_2 = (60\lambda b_1 + 7)(2Y^5 - 5Y^2) + 5(y^8 - 4Y^2) + \frac{\gamma-1}{\gamma}(20\lambda b_1 + 1)\frac{12}{t}[5(1-s)(Y^4 - 2Y^2) + 10s(2Y^3 - 3Y^2) - t(2Y^5 - 5Y^2)]$$

$$720\lambda^2\Theta_2 = (60\lambda b_1 + 7)(Y^4 - Y) + 4(Y^7 - Y) \tag{23}$$

etc., where from Eq. (10)

$$\lambda = 2\gamma t\mu_\infty/(\gamma+1)(\gamma-1)(1-t)\rho_\infty U_\infty \approx x/\xi \tag{24}$$

It is of interest to note that $\Theta_0 + \Theta_1x = F_0' + F_1'x$; the pressure gradient, although nonzero at the leading edge, makes no contribution to Eq. (12) up to $O(\xi)$. It remains to determine the constants a, b_1, b_2, etc., in the expansion for $\eta_s(x)$. If Eqs. (16–23) are substituted into Eq. (14), one obtains

$$a^2 = (\gamma+1)(\gamma-1)U_\infty\rho_\infty^2(3 + 3s - 2t)/6\gamma t \tag{25}$$

and b_1, b_2 are given by

$$\begin{aligned} \bar{p}_1(s,t) &= 4\lambda(3 + 3s - 2t)[b_1 + 2F_1(0)]/6 \\ &= -(9s + 2t)/180 \end{aligned} \tag{26}$$

$$\begin{aligned} \bar{p}_2(s,t) &= \lambda^2(3 + 3s - 2t)^2\{6[b_2 + 2b_1F_1(0) + 2F_2(0)] + 4[b_1 + 2F_1(0)^2]\}/36 \\ &= (1/388{,}800)\{(312 - 1092st - 220t^2 + 1944s + 1701s^2) - [6(\gamma-1)(9s + 2t)/\gamma t]\times (15 + 30s + 15s^2 - 3t - 4t^2 - 5st)\} \end{aligned} \tag{27}$$

Here $\bar{p}_1(s,t)$, $\bar{p}_2(s,t)$ are coefficients appearing in the series for the pressure

$$\frac{p(x)}{p_\infty} = \frac{\gamma-1}{6}M^2\frac{(3 + 3s - 2t)}{t}\left[1 + \bar{p}_1(s,t)\times\left(\frac{6\xi}{3 + 3s - 2t}\right) + \bar{p}_2(s,t)\left(\frac{6\xi}{3 + 3s - 2t}\right)^2 + \text{higher orders}\right] \tag{28}$$

The velocity in the viscous layer is

$$\frac{u}{U_\infty} = \frac{\partial F}{\partial Y} = Y + \frac{1}{12}(Y - Y^4)\xi + \frac{1}{120(3 + 3s - 2t)}\times \left\{\frac{1}{4}(8 + 5s - 6t)(Y^4 - Y) + \frac{4}{6}(3 + 3s - 2t)(Y^7 - Y) - \frac{\gamma-1}{\gamma}\frac{(9s + 2t)}{3t}[2(1-s)(Y^3 - Y) + 6s(Y^2 - Y) - t(Y^4 - Y)]\right\}\xi^2 + \text{higher orders} \tag{29}$$

The velocity $u = U_\infty Y$ in the limit $\xi \to 0$ corresponds precisely to the solution given by Oguchi, i.e.,

$$\begin{aligned} u/U_\infty &= [(1-s)/2] + [(1+s)/2]\sin[(\alpha + \pi/2)Y - \alpha] \\ \alpha &= \sin^{-1}[(1-s)/(1+s)] \end{aligned}$$

on the basis of which he calculated the pressure at the leading edge. The difference arises only because Oguchi assumed $\mu \sim (T)^{1/2}$. It should be noted that neither the unit Prandtl number nor the linear viscosity-temperature dependence constitutes assumptions essential to the solution given here. They are used only because the derivation is simplified greatly without compromise of the essentials.

In Oguchi's solution $\partial^2F/\partial Y^2 \to 0$ as $Y \to 1$, which is not the case here. This does not, however, imply zero shear stress at the shock, as Oguchi claims. In fact, the shear stress cannot be zero there, since the solution is obtained by stating that it is constant through the viscous layer. The misunderstanding appears to arise because $\rho\mu \to \infty$ if one uses the hypersonic approximation $2H \approx u^2$ at the edge of the viscous layer in conjunction with $\mu \sim T^s$ with $s < 1$. If the shock wave is treated as a discontinuity, the proper boundary conditions require that the shear stress at the shock wave correspond to the vorticity due to shock curvature. This condition cannot, in general, be imposed on a solution based on the boundary layer equations and is not met here in the region closest to the leading edge ($\xi \rightarrow 0$) where, moreover, the no-slip condition is invalid also. It may be noted that $(\mu\ \partial u/\partial y)_s \sim p(x)\ (\partial^2F/\partial Y^2)_s/(x)^{1/2}\eta_s(x)$ decreases rapidly with increasing ξ and falls to zero for $\xi \approx 4$ where, however, the convergence of the series for $(\partial^2F/\partial Y^2)_s$ is rather poor. An alternative treatment of the leading edge problem is given by Street,[3] who proposes to satisfy boundary conditions at infinity rather than at the shock wave, which appears to imply that his flow model is not one in which the viscous layer and the shock layer coincide.

Slip and temperature jump are negligible if $\xi \gg (\gamma + 1)s/\gamma$. For a cold wall in the hypersonic limit ($s \rightarrow 0$; $t = 1$), convergence of the series for pressure and velocity requires that $\xi < 10$ approximately. The solution given here is therefore valid in $h_w/H_\infty \ll Re/M^2 < 10\gamma/(\gamma + 1)$ or for $\chi \approx O(M^2)$.

References

[1] Oguchi, H., "The sharp leading edge problem in hypersonic flow," *Rarefied Gas Dynamics* (Academic Press Inc., New York, 1961), Sec. 5, pp. 501–524.

[2] Hayes, W. D. and Probstein, R. F., Hypersonic Flow Theory (Academic Press Inc., New York, 1959), pp. 335–338.

[3] Street, R. E., "Effect of slip on the laminar boundary layer near the leading edge of a flat plate in hypersonic rarefied gas flow," Boeing Flight Sci. Lab., Rep. 49 (November 1961).

Optimization of Stochastic Trajectories

L. J. Berman,* G. E. Steinker,† and R. Krupp‡
Massachusetts Institute of Technology, Cambridge, Mass.

Nomenclature

a_1, a_2, a_3 = weighting factors in payoff function
b = thrust per unit initial mass
$\mathbf{f}(\ )$ = vector function of ( )
$\mathbf{F}$ = matrix of partial derivatives of $\mathbf{f}$ with respect to $\mathbf{y}$, $F_{ij} = \partial f_i/\partial y_j$
g = gravitational acceleration
$\mathbf{G}$ = matrix of partial derivatives of $\mathbf{f}$ with respect to $\mathbf{z}$, $G_{il} = \partial f_i/\partial z_l$
$\mathbf{H}$ = measurement inference matrix; converts state vector to expected measurement vector
$\mathbf{K}$ = gain matrix for estimator
m = mass
$\mathbf{P}$ = covariance matrix of uncertainty in state vector
$\mathbf{Q}$ = covariance matrix of uncertainty in control vector
$\mathbf{R}$ = covariance matrix of uncertainty in measurement vector
t = time
u = horizontal velocity
v = vertical velocity
$\mathbf{X}$ = measurement vector
y = altitude
$\mathbf{y}$ = state vector $\{y,v,u\}$
$\mathbf{Y}$ = uncertainty in state vector
$\mathbf{z}$ = control vector $\{b,\theta\}$
$\mathbf{Z}$ = uncertainty in control vector
γ = fractional mass flow per second
δ = perturbation
Δ = increment
$\boldsymbol{\lambda}$ = matrix of adjoint variables
Λ = transition matrix for $\boldsymbol{\lambda}$
σ = standard deviation
ϕ = payoff quantity
θ = thrust angle, measured from horizontal

Subscripts

d = final descent phase
0 = initial (ignition) time
1 = final (end of main phase) time

Superscripts

$\cdot$ = time derivative
$'$ = matrix transpose
-1 = matrix inversion

Received by ARS November 30, 1962. This research was supported by the U. S. Air Force through the Office of Scientific Research of the Office of Aerospace Research, under Contract No. AF 49(638)-363. The work was performed under the supervision of R. L. Halfman of the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology.

* Research Engineer, Division of Aerospace Research, Member AIAA.

† Research Assistant, Division of Aerospace Research, Member AIAA.

‡ Mathematician, Division of Aerospace Research.

Introduction

In recent years, much study has gone into the problems of trajectory optimization[1–3] and the problem of optimal control around a reference trajectory.[4] The control analyses are linearized; hence their application to extremal trajectories gives indeterminate results. This has led to the formulation of second-order optimal control theory.[5] These analyses are based on deterministic situations; that is, the state of the system is known, and control changes can be applied exactly. Real problems, however, are not deterministic. The state only can be inferred from a possibly incomplete set of noisy measurements, and the control variables themselves are subject to random variation. This fact has led to the formulation of optimum linear filters.[6]§

Unfortunately, this acceptance of the true, stochastic nature of the problem always has come after the reference trajectory has been chosen. For some operational criteria this is correct, but if the criterion for choice is the optimality of the trajectory, and if the performance is affected by the statistics of the situation, the effect of the statistics should be incorporated into the optimization procedure. The problem cannot be discussed in generality but must be illustrated through a specific example.

As an example, the authors have examined the problem of achieving a soft landing on the moon, which was considered deterministically in Ref. 7. Such a trajectory very likely will consist of two phases: first a main phase during which most of the energy of the vehicle is dissipated, followed by a terminal phase in which the vehicle descends slowly to touchdown. This second phase is less efficient at energy dissipation than the main phase; hence it should begin at as low an energy level as is feasible. However, the transition point must be tied to the uncertainty in the altitude and velocity of the vehicle. Thus it is apparent that a main-phase trajectory that reduces uncertainties can reduce terminal phase propellant consumption and, in fact, can reduce the total expected propellant consumption if the main phase consumption is not affected seriously. A more detailed presentation of this work is given in Ref. 8.

Equations of Motion and the Filter

The vehicle is assumed to be a constant thrust rocket moving in a uniform, parallel gravity field, with no other external

§ In Ref. 6, Kalman has shown that the optimal linear control and the optimal linear filter are duals.